

ON THE ALEXANDER POLYNOMIAL OF LINKS IN LENS SPACES

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ABSTRACT. We explore properties of the Alexander polynomial and twisted Alexander polynomial of links in the lens spaces. In particular, we calculate the Alexander polynomial of some families of links and show how the Alexander polynomial is connected with the classical Alexander polynomial of the link in S^3 , obtained by cutting out the exceptional lens space fiber or replacing the exceptional fiber with a trivial one. As an application we show that the Alexander polynomial respects a skein relation.

1. INTRODUCTION & BACKGROUND

While the classical knot theory is a widely branched area of mathematical research, our knowledge about knots and links in other 3-manifolds still holds many questions. In this paper we will be particularly interested in the Alexander polynomial, which is perhaps one of the most extensively studied invariants in the classical knot theory. We now know that the Alexander polynomial holds information about sliceness, fiberedness, and the symmetries of knots.

The first definition of the Alexander polynomial was constructed from the homology of the infinite cyclic cover of the knot's complement in 1928 by Alexander [1], but it soon became clear that the polynomial can be studied from several different viewpoints. In 1962 Milnor observed that the polynomial can be defined through the Reidemeister torsion [17], and a few years later, in 1967, Conway showed that the polynomial is characterised using the local skein relations, although this approach was only popularized due to Kauffman [14]. Another notable construction, closely related to the Alexander's original idea, arises from the fundamental group of the knot's complement via Fox calculus [5].

In 1975 Turaev extended Milnor's idea and defined the Alexander polynomial for links in other 3-manifolds [21]. In 1990 Lin presented the idea of a twisted Alexander polynomial [16], which generalizes the Alexander polynomial in the sense that the Alexander polynomial is a special case of the twisted version and the polynomials are the same for a link in S^3 . While all the various definitions of the Alexander polynomial for a link in S^3 coincide, it was not until 2008 that Huynh and Le showed that the Alexander polynomial respects the skein relation for a link in the

Date: August 23, 2016.

2010 Mathematics Subject Classification. 57M27, 57M05 (primary), 57M25 (secondary).

Key words and phrases. links in lens spaces, Alexander polynomial.

projective plane [12]. In [2] an explicit presentation of the knot group in lens spaces is given.

While in [12] and [2] they use particular models of lens spaces, namely the “quotient of the 3-ball” model, in this paper we use a more general construction. By the Lickorish-Wallace Theorem [15, 23], any closed, orientable, connected 3-manifold can be obtained by performing Dehn surgery on a framed link in S^3 , furthermore, each component of the link can be assumed to be unknotted. Thus, we view the lens space $L(p, q)$ as the resulting space of a $-p/q$ rational surgery performed along the unknot in S^3 . This approach may be generalized and used for the links in other 3-manifolds.

Noticable related literature includes [4], where Drobotukhina defined a Jones-type polynomial for the projective space and also [19], where Mroczkowski calculated the HOMFLY-PT and Kauffman bracket skein modules (polynomial invariants defined exclusively by skein relations) of the projective space. In [18] the Kauffman bracket skein module was computed for a family of prism manifolds, and in [11] the Kauffman bracket skein module of lens spaces is computed. Recently, the much more complex HOMFLY-PT skein module was computed for lens spaces $L(p, 1)$ in [9]. In [8], a link group presentation for the links in arbitrary Seifert manifolds is given, but again a particular model of the space is used that cannot be generalized to arbitrary 3-manifolds.

This paper is organized as follows. In Section 2 we give an explicit construction for the presentation of the fundamental group of the complement of a link in a lens space and derive a formula for the first homology group. In Section 3 we present a definition of the (twisted) Alexander polynomial via the Alexander-Fox matrix and for the reader’s convenience present alternative proofs of some known results. In Section 4 the main results are given, namely the connection between the Alexander polynomial of a link in a lens space and of its classical counterpart in S^3 , or alternatively, the connection between a link in S^3 and the link obtained by performing a rational Dehn surgery on a trivial component. Finally, in Section 5 some sample calculations are done, and it is shown that the Alexander polynomial in lens spaces respects a skein relation.

2. THE PRESENTATION OF THE GROUP OF A LINK IN A 3-MANIFOLD

The Alexander polynomial of a link may be derived from a presentation of the link group using Fox calculus. A classical link L in the 3-sphere S^3 has a widely known Wirtinger presentation for $\pi_1(S^3 \setminus L, *)$. We begin this Section by recalling how to construct this presentation. Then we generalize this result by producing a presentation of the link group for the links in the lens spaces $L(p, q)$. Finally, we describe a presentation for the group of a link in any closed, connected, orientable 3-manifold M .

2.1. The group of a link in S^3 . Let us begin with the classical case, where L is a link in S^3 . Divide the 3-sphere S^3 into two 3-balls by a Heegaard decomposition of genus 0:

$$S^3 = B_{(1)}^3 \cup_{S^2} B_{(2)}^3 .$$

We think of the dividing surface S^2 as the 1-point compactification of the plane \mathbb{R}^2 , while the 3-balls $B_{(1)}^3$ and $B_{(2)}^3$ are represented by the upper and the lower half-space respectively. The diagram of the link L is drawn in the plane as a finite collection of arcs, which we call the *overpasses*. We may suppose that only the endpoints of each overpass lie on the sphere S^2 , while the rest of the overpass is included in the upper 3-ball $B_{(1)}^3$. We may assume that L does not intersect the center of $B_{(1)}^3$. Each crossing in the diagram contains two "lose" endpoints of the overpasses - these are the endpoints of an *underpass*. We suppose that each underpass is completely contained on the sphere S^2 . Thus, the link L is included in the upper 3-ball $B_{(1)}^3$, with the underpasses on the boundary S^2 .

We are going to compute the presentation of the link group $\pi_1(S^3 \setminus L, *)$ using the Seifert-van Kampen's Theorem. A similar construction may be found in [20, Chapter 3]. Let the basepoint $*$ lie in the center of $B_{(1)}^3$. Denote by $\alpha_1, \dots, \alpha_n$ the overpasses, and by β_1, \dots, β_n the underpasses of L . For $i = 1, \dots, n$, let x_i be a simple loop in $B_{(1)}^3$, based at $*$ and linking the overpass α_i with linking number 1, while not linking any other overpass. Denote by $A_1 = B_{(1)}^3 \setminus L$ the complement of the link in the upper 3-ball and let $A_2 = (B_{(2)}^3 \setminus L) \cup c$, where c is a path from a chosen point $b \in \partial B_{(2)}^3$ to the basepoint $*$ which does not intersect L .

Now A_1 is a 3-ball with n holes, drilled along the arcs $\alpha_1, \dots, \alpha_n$, whose boundary is a 2-sphere with n arcs β_1, \dots, β_n removed. Removing the arcs β_1, \dots, β_n does not change the homotopy type of the set in question, thus A_1 is homotopically equivalent to the bouquet of n loops x_1, \dots, x_n and $\pi_1(A_1, *) = \langle x_1, \dots, x_n \rangle$.

Since the link L is contained in the upper 3-ball $B_{(1)}^3$, the set A_2 is a 3-ball (together with the path c) on whose boundary n arcs β_1, \dots, β_n are removed. Thus, A_2 is simply connected.

The intersection $A_1 \cap A_2$ is a 2-sphere (together with the path c) with n disjoint arcs β_1, \dots, β_n missing. For $i = 1, \dots, n$, choose a simple loop $\hat{y}_i \subset A_1 \cap A_2$, based at the point b and encircling the arc β_i . Thus, $\pi_1(A_1 \cap A_2, *) = \langle y_1, \dots, y_n \rangle$ is a free group, whose i -th generator is the loop $y_i = \bar{c} \cdot \hat{y}_i \cdot c$ for $i = 1, \dots, n$. Each underpass β_i is adjacent to two overpasses α_{i_1} and α_{i_2} and crossed by the third overpass α_{i_3} .

Definition 2.1. Let L be an oriented link (in the 3-sphere or in a lens space), given by a plane diagram. A chosen crossing of L in the diagram is called **positive** if the pair of orientations (orientation of the overpass, orientation of the underpass) defines the positive orientation of the plane of the diagram (Figure 1(a)), otherwise it is called **negative** (Figure 1(b)).

It is clear from the Figure 1 that the loop y_i encircling the underpass β_i is, when included in A_1 , either equivalent to the product $x_{i_1}x_{i_3}x_{i_2}^{-1}x_{i_3}^{-1}$ if the corresponding crossing is positive, or to the product $x_{i_1}x_{i_3}^{-1}x_{i_2}^{-1}x_{i_3}$ if the corresponding crossing is negative.

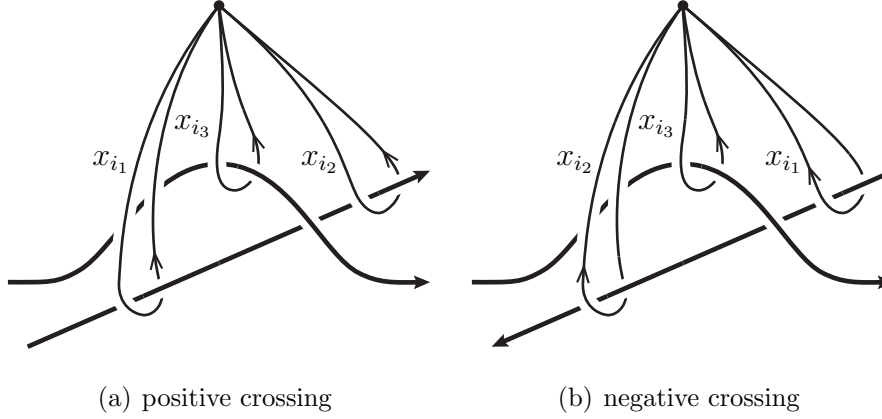


FIGURE 1. Wirtinger relations.

Applying the Seifert-van Kampen's Theorem, we obtain the Wirtinger presentation for $\pi_1(S^3 \setminus L, *) = \pi_1(A_1 \cup A_2, *)$:

Theorem 2.2 ([20]). *Let $L \subset S^3$ be a link in the 3-sphere, given by a plane diagram. Using the notation introduced above, the group of the link L has a presentation*

$$\pi_1(S^3 \setminus L, *) = \langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle ,$$

where r_i for $i = 1, \dots, n$ is the Wirtinger relation $x_{i_1}x_{i_3}x_{i_2}^{-1}x_{i_3}^{-1}$ or $x_{i_1}x_{i_3}^{-1}x_{i_2}^{-1}x_{i_3}$, corresponding to the i -th underpass β_i of the link L .

Remark 2.3. *We may also construct this presentation using a slightly different approach. Let us build the manifold $S^3 \setminus L$ as a CW-complex. As we have seen, the link L is contained in the upper 3-ball $B_{(1)}^3$, and the complement $B_{(1)}^3 \setminus L$ is homotopically equivalent to a bouquet of n loops x_1, \dots, x_n . Thus, we start with the basepoint $*$ representing the 0-cell, and add the 1-cells x_1, \dots, x_n . Now we need to complete the boundary of a regular neighbourhood of every underpass. To do this, we add a 2-cell, whose boundary encompasses the underpass and is either equivalent to the product $x_{i_1}x_{i_3}x_{i_2}^{-1}x_{i_3}^{-1}$ or to $x_{i_1}x_{i_3}^{-1}x_{i_2}^{-1}x_{i_3}$. The resulting CW-complex is homotopically equivalent to $S^3 \setminus L$ and its fundamental group has the presentation*

$$\langle x_1, \dots, x_n \mid r_1, \dots, r_n \rangle ,$$

same as above.

Corollary 2.4 ([20]). *Let $L \subset S^3$ be a link with r components. Then the first homology group of the link complement equals $H_1(S^3 \setminus L) \cong \mathbb{Z}^r$.*

Proof. We abelianize the link group $\pi_1(S^3 \setminus L, *)$ with the Wirtinger presentation, given by the Theorem 2.2. If the generators x_1, \dots, x_n commute, then the Wirtinger relations imply that all the generators, corresponding to the same link component of L , become homologous. Thus, $H_1(S^3 \setminus L)$ is the free abelian group on r generators. The generator of the i -th \mathbb{Z} -summand corresponds to the meridian of the regular neighbourhood of the i -th component of L . \square

2.2. The group of a link in a lens space. Let p and q be coprime integers. The lens space $L(p, q)$ may be constructed as follows. Describe the 3-sphere S^3 as a union of two solid tori V_1 and V_2 (the Heegaard decomposition of genus 1). Choose a meridian m_1 and a longitude l_1 , generating $\pi_1(\partial V_1)$. Then the lens space $L(p, q)$ is obtained from S^3 by the $-p/q$ surgery on the second solid torus V_2 . Thus we remove V_2 from S^3 and then glue it back onto V_1 by a homeomorphism $h: \partial V_2 \rightarrow \partial V_1$, which maps the meridian m_2 of V_2 by

$$h_*(m_2) = pl_1 - qm_1,$$

see Figure 2.

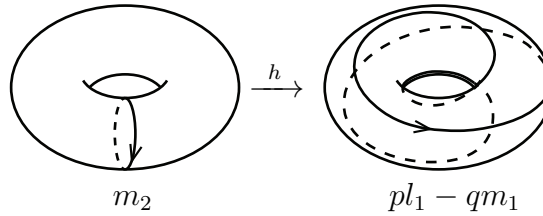


FIGURE 2. Heegaard decomposition of $L(p, q)$.

This description is easy to see in the Kirby diagram of $L(p, q)$, presented in Figure 3(a), i.e. the unknot U bearing the framing $-p/q$ (see [10] as a good reference on Kirby diagrams). The unknot U represents the meridian m_1 , on whose regular neighbourhood the surgery is performed. Thus, the lens space $L(p, q)$ is completely defined by three data: p , q and the position of the meridian m_1 .

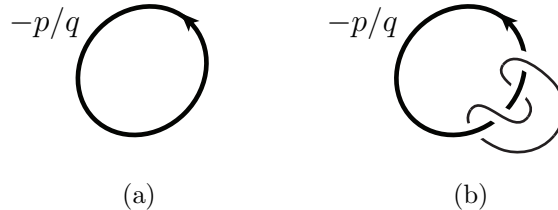


FIGURE 3. The Kirby diagram for $L(p, q)$ and diagram of a knot in $L(p, q)$.

Now consider a link L in the lens space $L(p, q)$. Bearing in mind the Heegaard decomposition of $L(p, q)$ into two solid tori V_1 and V_2 , we may assume that the link L is contained in the first solid torus V_1 . Represent $L(p, q)$ by its Kirby diagram

as an unknot U bearing the framing $-p/q$, and draw L inside this diagram, see Figure 3(b). Such diagrams are also called mixed link diagrams, see for example [3].

The following generalized Reidemeister theorem tells us when two diagrams represent isotopic links.

Theorem 2.5 ([11, 18]). *Two links L_1 and L_2 in $L(p, q)$ with diagrams D_1 and D_2 are isotopic if and only if there exists a finite sequence of moves Ω_1 , Ω_2 , Ω_3 , and $SL_{p,q}$, depicted in Figures 4 and 5, that transform one diagram to the other.*

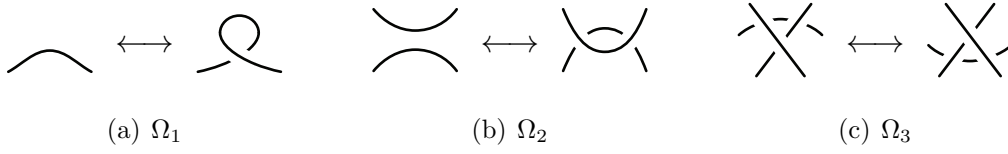


FIGURE 4. Classical Reidemeister moves.

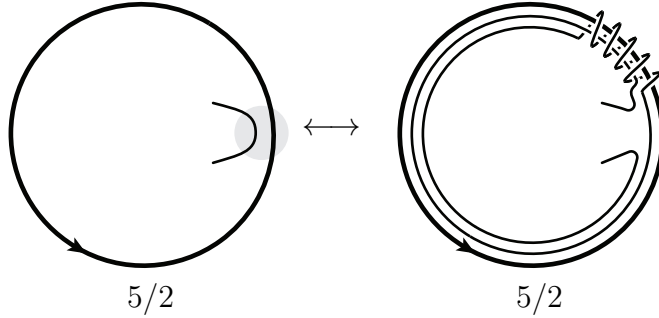


FIGURE 5. The slide move $SL_{5,2}$.

The move $SL_{p,q}$ from the theorem above is the isotopy taking the connected sum of a link $L \subset L(p, q)$ with the boundary of the 2-cell, defining the lens space $L(p, q)$, which is, on the diagram, a $K_{p,q}$ knot that winds p times around the longitude and q times along the meridian of the regular neighbourhood of $S^3 \setminus U$.

To obtain the complement $L(p, q) \setminus L$, we first remove the regular neighbourhood of L in V_1 and then perform surgery on U . This means we actually take the complement of the union $L \cup U$ in S^3 and then glue in the solid torus V_2 according to the homeomorphism $h: \partial V_2 \rightarrow \partial V_1$.

The presentation of the link group $\pi_1(L(p, q) \setminus L)$ may be obtained from the decomposition

$$L(p, q) \setminus L = (V_1 \cup_h V_2) \setminus L = (V_1 \setminus L) \cup_h V_2 = (S^3 \setminus (L \cup U)) \cup_h V_2.$$

The first set $A = S^3 \setminus (L \cup U)$ is the complement of a link in S^3 and its fundamental group has a classical Wirtinger presentation, described in Subsection 2.1.

The second set is a solid torus V_2 , which maps onto A along ∂V_1 . The fundamental group $\pi_1(\partial V_1) = \mathbb{Z} \oplus \mathbb{Z}$ is generated by the longitude l_1 and the meridian m_1 . The solid torus V_2 is mapped onto $\partial V_1 \subset \partial A$ in two parts: firstly, the cylinder $B^2 \times I$ is mapped as a 2-cell with boundary $m_2 \times I$ according to the map $h_*(m_2 \times \{0\}) = p \cdot l_1 - q \cdot m_1$, then the remaining 3-cell $B^2 \times I \cong B^3$ is mapped onto the remaining 2-sphere boundary. Adding the 2-cell introduces a new relation $l_1^p \cdot m_1^{-q} = 1$ in the fundamental group. Thus, the presentation for $\pi_1(L(p, q) \setminus L, *)$ is obtained from the presentation for $\pi_1(S^3 \setminus (L \cup U), *)$ by adding the **lens relation** $l_1^p \cdot m_1^{-q}$.

Definition 2.6. A link $L \subset L(p, q)$ is **affine** (sometimes also called *local*), if it is contained in a 3-ball $B^3 \subset L(p, q)$.

The lens relation, of course, has to be written in terms of the Wirtinger generators of $\pi_1(S^3 \setminus (L \cup U), *)$. Since l_1 is the longitude of the regular neighbourhood of $S^3 \setminus U$, it follows that any Wirtinger generator, corresponding to the unknot U , represents l_1 . The meridian m_1 represents the homotopy class of U itself. If L is an affine link in $L(p, q)$, then U may be homotoped to a point and thus m_1 is trivial in $\pi_1(S^3 \setminus (L \cup U), *)$. If L is not affine, then the meridian m_1 may be expressed in terms of the Wirtinger generators, corresponding to L , as will be explained below. We have obtained the following result:

Theorem 2.7. Let L be a link in the lens space $L(p, q)$. Represent $L(p, q)$ by an unknot U bearing the framing $-p/q$, and draw L inside this diagram. Let $\langle x_1, \dots, x_n \mid w_1, \dots, w_n \rangle$ be the Wirtinger presentation for $\pi_1(S^3 \setminus (L \cup U), *)$, obtained from this diagram. Denote by m_1 and l_1 the meridian and longitude of the regular neighbourhood of $S^3 \setminus U$, written in terms of the generators x_1, \dots, x_n . Then the presentation for the link group $\pi_1(L(p, q) \setminus L, *)$ is given by

$$\langle x_1, \dots, x_n \mid w_1, \dots, w_n, m_1^p \cdot l_1^{-q} \rangle .$$

The above construction may also be generalized to yield the link group of a link inside any closed, connected, orientable 3-manifold M .

Theorem 2.8. Let L be a link in a closed connected orientable 3-manifold M . Represent M as the result of an integer surgery on a k -component framed link $L_0 \subset S^3$ and draw the Kirby diagram of $L \cup L_0$. Let $\langle x_1, \dots, x_n \mid w_1, \dots, w_n \rangle$ be the Wirtinger presentation of $\pi_1(S^3 \setminus (L \cup L_0), *)$. For the i -th component of L_0 , let p_i be its framing and denote by m_i and l_i the meridian and longitude of the regular neighbourhood of its complement in S^3 , written in terms of the generators x_1, \dots, x_n . Then the group of the link L is given by the presentation

$$\pi_1(M \setminus L, *) = \langle x_1, \dots, x_n \mid w_1, \dots, w_n, m_1^{p_1} \cdot l_1^{-1}, \dots, m_k^{p_k} \cdot l_k^{-1} \rangle .$$

Starting with the diagram of $L \subset L(p, q)$, we will now introduce a notation for the generators and relations of the link group to make the calculations easier to follow.

Notation 2.9. Let $L \subset L(p, q)$ be an oriented link. Represent $L(p, q)$ by its Kirby diagram as the $-p/q$ surgery on an oriented unknot U and draw L inside this diagram. As we have shown in the Theorem 2.7, the presentation of the link group of L is obtained from the presentation of the link group $\pi_1(S^3 \setminus (L \cup U), *)$ by adding one relation. The Wirtinger generators, corresponding to the link L , will be denoted by x_i , while the generators, corresponding to the unknot U , will be denoted by a_j .

Let D be the obvious disk in S^3 that is bounded by U . Take a small cillinder $C = D \times [-\epsilon, \epsilon]$ that is a regular neighbourhood of D in S^3 . If L is an affine link, then we may assume that L does not cross C , and denote by x_1, \dots, x_n the generators of $\pi_1(L(p, q) \setminus L, *)$, corresponding to L , and by a_1 the generator of $\pi_1(L(p, q) \setminus L, *)$, corresponding to U .

If L is not affine, we may assume that $L \cap C$ is a union of k parallel strands s_1, \dots, s_k . Each of the strands is overcrossed by the unknot U , which divides s_i into two arcs $\bar{\alpha}_i \subset D \times [0, \epsilon]$ and $\bar{\alpha}_{k+i} \subset D \times [-\epsilon, 0]$ (for $i = 1, \dots, k$). The arc $\bar{\alpha}_i$ is a part of the overpass α_i in the diagram, and we denote by x_i its corresponding generator in $\pi_1(L(p, q) \setminus L, *)$ for $i = 1, \dots, 2k$. It may happen that the arcs α_i and α_j for two different indices $1 \leq i, j \leq 2k$ coincide; in this case we add the relation $x_i = x_j$ to the presentation of $\pi_1(L(p, q) \setminus L, *)$. We denote by a_1 the generator of $\pi_1(L(p, q) \setminus L, *)$, corresponding to the overpass of the unknot U which overcrosses the strands s_1, \dots, s_k . Moreover, we denote by a_i the generator of $\pi_1(L(p, q) \setminus L, *)$, corresponding to the arc of U which lies between the overcrossings of s_{k-i+1} and s_{k-i+2} with U for $i = 2, \dots, k$.

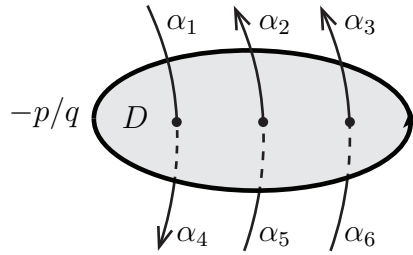


FIGURE 6. Intersections of strands with the disk for $k = 3, \epsilon_1 = -1, \epsilon_2 = 1, \epsilon_3 = 1$.

The orientation of L induces an orientation for each of the strands s_1, \dots, s_k . For $i = 1, \dots, k$, let $\epsilon_i = 1$ if the strand s_i is oriented so that the arc $\bar{\alpha}_i$ comes after the arc $\bar{\alpha}_{k+i}$, and let $\epsilon_i = -1$ otherwise.

Corollary 2.10. In this notation, the presentation of the link group is given by

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1, \dots, a_k \mid w_1, \dots, w_{n+k}, a_1^p (x_1^{\epsilon_1} \dots x_k^{\epsilon_k})^{-q} \rangle,$$

in particular, if L is affine, we have

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1 \mid w_1, \dots, w_n, a_1^p \rangle.$$

Proof. It follows directly from the Theorem 2.7 and the introduced notation. \square

Lemma 2.11. *Any link $L \subset L(p, q)$ with the diagram, described in 2.9, is equivalent to a link whose algebraic intersection with the disk D is an integer between 0 and $p - 1$.*

Proof. The algebraic intersection of L with the disk D equals $\sum_{i=1}^k \epsilon_i$, while the algebraic intersection of $SL^r(L)$ with D equals $\sum_{i=1}^k \epsilon_i + r p$ for $r \in \mathbb{Z}$. It follows that L is equivalent to a link for which $0 \leq \sum_{i=1}^k \epsilon_i \leq p - 1$. \square

From now on, if not stated otherwise, we will assume that any link $L \subset L(p, q)$ has $0 \leq \sum_{i=1}^k \epsilon_i \leq p - 1$, as we may by the Lemma 2.11.

To end this Section, we describe the abelianization of the link group of L . The following Lemma and Corollary are rewritten from [2, Lemma 4, Corollary 5]:

Lemma 2.12 ([2]). *Let $K \subset L(p, q)$ be an oriented knot. Represent $L(p, q)$ as the $-p/q$ surgery on an unknot U , and draw K inside this diagram. Let D be the obvious disk in S^3 , bounded by U . Denote by $f(K)$ the algebraic intersection number of K and D . Then the homology class of K inside $H_1(L(p, q)) \cong \mathbb{Z}_p$ is equal to*

$$[K] = q \cdot f(K) \text{ mod } p .$$

Proof. The lens space $L(p, q)$ is obtained from S^3 by removing the regular neighbourhood νU of U and glueing in a 2-cell with the boundary $p \cdot l_1 - q \cdot m_1$, where $l_1 \in \pi_1(\partial(\nu U))$ is the meridian of νU and $m_1 \in \pi_1(\partial(\nu U))$ is the longitude of U . For each positive (negative) point of intersection between K and D , there are q positive (negative) points of intersection between K and the 2-cell of $L(p, q)$. By summing up all the signed intersections, the above formula is obtained. \square

Corollary 2.13 ([2]). *Let $L \subset L(p, q)$ be a link with components L_1, \dots, L_r . Denote by c_i the homology class of L_i in $H_1(L(p, q)) \cong \mathbb{Z}_p$ for $i = 1, \dots, r$. The first homology group of the link complement equals*

$$H_1(L(p, q) \setminus L) \cong \mathbb{Z}^r \oplus \mathbb{Z}_d ,$$

where $d = \gcd(c_1, \dots, c_r, p)$.

Proof. Represent $L(p, q)$ by the Kirby diagram as the $-p/q$ surgery on an unknot U , and draw L inside this diagram. Let D be the obvious disk in S^3 , bounded by U . For $i = 1, \dots, r$ denote by $f(L_i)$ the algebraic intersection number of L_i and D . We abelianize the link group $\pi_1(L(p, q) \setminus L, *)$, whose presentation is given by the Corollary 2.10. Once the generators commute, then by the Wirtinger relations w_1, \dots, w_{n+k} all the generators, corresponding to the same link component of L , become homologous. Thus, $H_1(L(p, q) \setminus L)$ is an abelian group with r free generators g_1, \dots, g_r , corresponding to the components of the link L , and the generator a_1 , corresponding to the unknot U . The lens relation, when abelianized, becomes

$$p \cdot a_1 + q \cdot (f(L_1)g_1 + \dots + f(L_r)g_r) = 0 .$$

By the Lemma 2.12, this is the same as $p \cdot a_1 + (c_1 g_1 + \dots + c_r g_r) = 0$, and it follows that $d = \gcd(c_1, \dots, c_r, p)$. \square

3. THE ALEXANDER-FOX MATRIX OF A LINK IN THE LENS SPACE

3.1. The construction and definitions. Given a presentation of the group of a link, one may calculate its Alexander polynomial using the Fox calculus. We shortly recall the following construction from [22]. Let

$$\mathcal{P} = \langle x_1, \dots, x_n \mid r_1, \dots, r_m \rangle$$

be a presentation of a group G and denote by $H = G/G'$ its abelianization. Let $F = \langle x_1, \dots, x_n \rangle$ be the corresponding free group. We apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H,$$

where $\frac{\partial}{\partial x}$ denotes the Fox differential, γ is the quotient map by the relations r_1, \dots, r_m and α is the abelianization map. The *Alexander-Fox matrix* of the presentation \mathcal{P} is the matrix $A = [a_{i,j}]$, where $a_{i,j} = \alpha(\gamma(\frac{\partial r_i}{\partial x_j}))$ for $i = 1, \dots, m$ and $j = 1, \dots, n$. For $k = 1, \dots, \min\{m-1, n-1\}$, the *k-th elementary ideal* $E_k(\mathcal{P})$ is the ideal of $\mathbb{Z}H$, generated by the determinants of all the $(n-k)$ minors of A . The *first elementary ideal* $E_1(\mathcal{P})$ is the ideal of $\mathbb{Z}H$, generated by the determinants of all the $(n-1)$ minors of A .

Definition 3.1. Let $L \subset S^3$ be a link, and let $E_k(\mathcal{P})$ be the *k-th elementary ideal*, obtained from a presentation \mathcal{P} of $\pi_1(S^3 \setminus L, *)$. Then the *k-th link polynomial* $\Delta_k(L)$ is the generator of the smallest principal ideal containing $E_k(\mathcal{P})$. The *Alexander polynomial* of L , denoted by $\Delta(L)$, is the first link polynomial of L .

For a classical link $L \subset S^3$, the abelianization of $\pi_1(S^3 \setminus L, *)$ is the free abelian group, whose generators correspond to the components of L , see Corollary 2.4. For a link in the lens space $L(p, q)$, the abelianization of its link group may also contain torsion, as we know by the Corollary 2.13. In this case, we need the notion of a twisted Alexander polynomial. We recall the following from [2].

Let G be a group with a finite presentation \mathcal{P} and abelianization $H = G/G'$ and denote $K = H/\text{Tors}(H)$. Then every homomorphism $\sigma: \text{Tors}(H) \rightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$ determines a twisted Alexander polynomial $\Delta^\sigma(\mathcal{P})$ as follows. Choosing a splitting $H = \text{Tors}(H) \times K$, σ defines a ring homomorphism $\sigma: \mathbb{Z}[H] \rightarrow \mathbb{C}[G]$ sending $(f, g) \in \text{Tors}(H) \times K$ to $\sigma(f)g$. Thus we apply the chain of maps

$$\mathbb{Z}F \xrightarrow{\frac{\partial}{\partial x}} \mathbb{Z}F \xrightarrow{\gamma} \mathbb{Z}G \xrightarrow{\alpha} \mathbb{Z}H \xrightarrow{\sigma} \mathbb{C}[G]$$

and obtain the σ -twisted Alexander matrix $A^\sigma = \left[\sigma(\alpha(\gamma(\frac{\partial r_i}{\partial x_j}))) \right]$. The twisted Alexander polynomial is then defined by $\Delta^\sigma(\mathcal{P}) = \gcd(\sigma(E_1(\mathcal{P})))$.

Definition 3.2. Let $L \subset L(p, q)$ be a link in the lens space. For any presentation \mathcal{P} of the link group $\pi_1(L(p, q) \setminus L, *)$, we may define the following.

The **Alexander polynomial** of L , denoted by $\Delta(L)$, is the generator of the smallest principal ideal containing $E_1(\mathcal{P})$.

For any homomorphism $\sigma: \text{Tors}(H_1(L(p, q) \setminus L)) \rightarrow \mathbb{C}^*$, the **σ -twisted Alexander polynomial** of L is $\Delta^\sigma(L) = \gcd(\sigma(E_1(\mathcal{P})))$.

We know from the Corollary 2.13 that the torsion subgroup of $H_1(L(p, q) \setminus L)$ is the group \mathbb{Z}_d . Thus, the image of the group homomorphism $\sigma: \text{Tors}(H_1(L(p, q) \setminus L)) \rightarrow \mathbb{C}^*$ is contained in the cyclic group, generated by the d -th root of the unity. When the chosen generator gen of $\text{Tors}(H_1(L(p, q) \setminus L))$ will be clear from the context, we will denote by $\Delta^\mu(L)$ the σ -twisted Alexander polynomial, for which $\sigma(gen) = \mu \in \mathbb{C}^*$.

3.2. The calculation of the Alexander-Fox matrix. In this subsection we will use the presentation of the link group $\pi_1(L(p, q) \setminus L, *)$ to calculate the Fox differentials and obtain the Alexander-Fox matrix A_L .

Let $L \subset L(p, q)$ be a link, given by a diagram, described in 2.9. The diagram of L consists of two parts: the first part is contained in the cylinder C , and the second part is the diagram in the rest of S^3 . The first part of the diagram determines the Wirtinger relations, corresponding to the crossings between L and U , and the lens relation. The second part of the diagram determines the Wirtinger relations, corresponding to the crossings within the link L . Using the notation, described in 2.9, we will now write down the relations, determined by the first part of the diagram.

If L is an affine link, then the first part of the diagram consists merely of the unknot U , determining a single relation $l: a_1^p$.

If L is not affine, then the first part of the diagram contains $2k$ crossings. By the Lemma 2.11 we may assume that $0 \leq \sum_{i=1}^k \epsilon_i \leq p - 1$. For $i = 1, \dots, k$, there is a crossing where U overcrosses the strand s_i , yielding the Wirtinger relation $q_i: a_1 x_{k+i}^{\epsilon_i} a_1^{-1} x_i^{-\epsilon_i}$. Moreover, for $i = 1, \dots, k$ there is a crossing where the strand s_i overcrosses U , yielding the Wirtinger relation $r_i: x_i^{\epsilon_i} a_{k-i+1} x_i^{-\epsilon_i} a_{k-i+2(\text{mod } k)}^{-1}$. Finally, the lens relation is $l: a_1^p (x_1^{\epsilon_1} \dots x_k^{\epsilon_k})^{-q}$. The presentation of the link group looks like

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k, w_1, \dots, w_{n-k}, l \rangle,$$

where w_1, \dots, w_{n-k} denote the Wirtinger relations, corresponding to the crossings within the link L .

Now we calculate the Fox differentials of the known relations, apply the quotient map γ by the relations and abelianize to obtain the Alexander-Fox matrix. Since the lens relation is essentially different from the other, Wirtinger relations, this process is done in two steps, as will be described in the following Proposition.

Definition 3.3. For a link $L \subset L(p, q)$, given by the diagram, described in 2.9, we denote $\bar{k} = \sum_{i=1}^k \epsilon_i$, $p' = \frac{p}{d}$ and $k' = \frac{\bar{k}}{d}$, where $d = \gcd\{p, \bar{k}\}$.

Proposition 3.4. *For a link $L \subset L(p, q)$, let*

$$\mathcal{P} = \langle x_1, \dots, x_n, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k, w_1, \dots, w_{n-k}, l \rangle$$

be the link group presentation, described above. Let M be the matrix of the Fox differentials of \mathcal{P} , and denote by $A(x, a)$ the matrix we obtain from M by identifying $x_i = x$ for $i = 1 \dots, n$ and $a_j = a$ for $j = 1, \dots, k$. Then the Alexander-Fox matrix of L is given by

$$A_L(t) = A(t^{\frac{p}{d}}, t^{\frac{q\bar{k}}{d}}) = A(t^{p'}, t^{qk'}) ,$$

while the μ -twisted Alexander-Fox matrix is given by $A_L^\mu(t) = A(t^{p'}, \nu t^{qk'})$, where $\mu \in \mathbb{C}^$ is a d -th root of unity and $\nu^{p'} = \mu$.*

Proof. As the Fox differentials of all the relations in the given presentation are calculated, we apply on them the quotient map γ by the relations, followed by the abelianization α . When applying the homomorphism $\alpha \circ \gamma$ with respect to the Wirtinger relations, all the generators, corresponding to the same component of the link $L \cup U$, become identified. We thus identify $\alpha(\gamma(a_1)) = \dots = \alpha(\gamma(a_k)) = a$. Since we will calculate the one-variable Alexander polynomial of L , we also identify $\alpha(\gamma(x_i)) = x$ for $i = 1 \dots, n$. By applying $\alpha \circ \gamma$ with respect to all relations except the lens relation, we therefore obtain the two-variable matrix $A(x, a)$, with x corresponding to the link L and a corresponding to the unknot U .

The Alexander-Fox matrix of L is calculated from $A(x, a)$ by applying $\alpha \circ \gamma$ with respect to the lens relation and thus identifying

$$a^p = \alpha(\gamma(a_1))^p = \alpha(\gamma(a_1^p)) = \alpha(\gamma((x_1^{\epsilon_1} \dots x_k^{\epsilon_k})^q)) = x^{q\bar{k}} .$$

Therefore, the Alexander-Fox matrix A_L of the link L is obtained from $A(x, a)$ by the substitution $A_L(t) = A(t^{\frac{p}{d}}, t^{\frac{q\bar{k}}{d}}) = A(t^{p'}, t^{qk'})$.

By the Corollary 2.13, the torsion of the abelianized link group is the group \mathbb{Z}_d . If there is nontrivial torsion, then the lens relation becomes $(a^{p'} x^{-qk'})^d = 1$ and the homomorphism $\sigma: \mathbb{Z}H \rightarrow \mathbb{C}[G]$ sends the torsion generator $a^{p'} x^{-qk'}$ to μ . It follows that the μ -twisted Alexander-Fox matrix A_L^μ is obtained from $A(x, a)$ by the substitution $A_L^\mu(t) = A(t^{p'}, \nu t^{qk'})$. \square

Remark 3.5. *Observe that \bar{k} is actually the algebraic intersection number of L and the disk D . If $L \subset L(p, q)$ is a link with the components L_1, \dots, L_r , then by the Lemma 2.12 we have $\sum_{i=1}^r [L_i] = (q\bar{k}) \pmod{p} \in H_1(L(p, q))$. Since p and q are coprime, it follows that $d = \gcd\{p, \bar{k}\} = \gcd\{p, q\bar{k}\} = \gcd\{p, \sum_{i=1}^r [L_i]\}$ and thus the number p' may be more invariantly defined as $p' = \frac{p}{\gcd\{p, \sum_{i=1}^r [L_i]\}}$.*

We calculate the matrix $A(x, a) =$

(1)

$$\begin{pmatrix} x_n \\ \vdots \\ x_{2k+1} \\ x_{2k} \\ \vdots \\ x_{k+1} \\ x_k \\ \vdots \\ x_1 \\ a_1 \\ \vdots \\ a_k \end{pmatrix} \begin{pmatrix} w_1 & \dots & w_{n-k} & q_k & \dots & q_1 & r_k & \dots & r_1 & l \\ & B_1 & & & 0 & & & 0 & & 0 \\ & & & \phi_k a & & & & & & \\ & B_3 & & & \ddots & & & 0 & & 0 \\ & & & & & \phi_1 a & & & & \\ & & & -\phi_k & & & \phi_k(1-a) & & & \beta_k \\ & B_2 & & & \ddots & & & \ddots & & \vdots \\ & & & & & -\phi_1 & & & \phi_1(1-a) & \beta_1 \\ & & & \phi_k(1-x) & \dots & \phi_1(1-x) & x^{\epsilon_k} & & -1 & \alpha \\ & 0 & & & 0 & & -1 & \ddots & & 0 \\ & & & & & & & -1 & x^{\epsilon_1} & \end{pmatrix}$$

where $\alpha = 1 + a + \dots + a^{p-1}$,

$$\beta_i = \begin{cases} -(1 + x^{\bar{k}} + \dots + x^{(q-1)\bar{k}})x^{\sum_{j=1}^{i-1} \epsilon_j} & \text{if } \epsilon_i = 1, \\ (1 + x^{\bar{k}} + \dots + x^{(q-1)\bar{k}})x^{\sum_{j=1}^i \epsilon_j} & \text{if } \epsilon_i = -1. \end{cases}$$

and

$$\phi_i = \begin{cases} 1 & \text{if } \epsilon_i = 1, \\ -x^{-1} & \text{if } \epsilon_i = -1. \end{cases}$$

for $i = 1, \dots, k$.

4. THE MAIN RESULTS

In this Section, we describe a relation between the Alexander polynomials of a link in the lens space and of its classical counterpart in S^3 . Let $L \subset L(p, q)$ be a link, given by a diagram, described in 2.9. Denote by $L' \subset S^3$ the classical link we obtain from L by ignoring the surgery along the unknot U . We will study the relation between the Alexander polynomial $\Delta(L)$ of $L \subset L(p, q)$ and the Alexander polynomials of the classical links $\Delta(L')$ and $\Delta(L' \cup U)$. Firstly, we observe the case of an affine link in the lens space. The following Corollary may be compared to the [2, Proposition 7].

Corollary 4.1. *Let $L \subset L(p, q)$ be an affine link. Then for its twisted Alexander polynomials we have $\Delta^1(L) = p \cdot \Delta(L')$ and $\Delta^\mu(L) = 0$ for $\mu \neq 1$.*

Proof. By the Corollary 2.10, the link group of an affine link has a presentation

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1 \mid w_1, \dots, w_n, a_1^p \rangle .$$

Denoting by A_L and $A_{L'}$ the Alexander-Fox matrices for L and L' respectively, A_L equals $A_{L'}$ with one additional row (corresponding to a_1) and column (corresponding to the lens relation $l: a_1^p$). From the lens relation it follows that the abelianization of $\pi_1(L(p, q) \setminus L, *)$ has the torsion subgroup \mathbb{Z}_p . If $\sigma: \mathbb{Z}_p \rightarrow \mathbb{C}^*$ is the homomorphism, which takes the generator a_1 to $\mu \neq 1$, then we have $\sigma(\alpha(\gamma(\frac{\partial l}{\partial a_1}))) = \frac{\mu^p - 1}{\mu - 1} = 0$.

For the homomorphism $\sigma: \mathbb{Z}_p \rightarrow \mathbb{C}^*$ which takes the generator a_1 to 1, we have $\sigma(\alpha(\gamma(\frac{\partial l}{\partial a_1}))) = p$, and this is the only nonzero entry in the last row (and column) of the matrix A_L . For the minors of A_L , we observe the following. Since the Wirtinger presentation of a classical link group has deficiency one, we have $\det A_L^{n+1, n+1} = \det A_{L'} = 0$. Moreover, if we remove from A_L either any row and the last column or the last row and any column, then the remaining matrix is singular. It follows that

$$\det A_L^{i,j} = \begin{cases} p \cdot \det A_{L'}^{i,j} & \text{if } 1 \leq i, j \leq n, \\ 0 & \text{if } i = n+1 \text{ or } j = n+1, \end{cases}$$

and we conclude

$$\gcd\{\det A_L^{i,j} \mid 1 \leq i, j \leq n+1\} = p \cdot \gcd\{\det A_{L'}^{i,j} \mid 1 \leq i, j \leq n\} .$$

□

Now we explore the case of a link $L \subset L(p, q)$ which is not affine. In this case, L is nontrivially linked with the unknot U . We observe that the Alexander polynomial of $L \subset L(p, q)$ is related to the Alexander polynomial of the classical link $L' \cup U \subset S^3$:

Proposition 4.2. *Let $\Delta(L)$ be the Alexander polynomial of $L \subset L(p, q)$. Denote by $\Delta_i(L' \cup U)$ the i -th two-variable polynomial of the classical link $L' \cup U \subset S^3$ (the variables corresponding to L' and U respectively). Then $\Delta(L)(t)$ divides $\Delta_1(L' \cup U)(t^{p'}, t^{q'})$ and is divisible by $\Delta_2(L' \cup U)(t^{p'}, t^{q'})$.*

Proof. The presentation

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k, w_1, \dots, w_{n-k}, l \rangle ,$$

described in Subsection 3.2, is obtained from the presentation for $\pi_1(S^3 \setminus (L' \cup U), *)$ by adding the lens relation, see Theorem 2.7. Applying $\alpha \circ \gamma$ with respect to all relations except the lens relation means identifying all the variables x_i and all the variables a_i in the Fox differentials of this presentation. We obtain the matrix $A(x, a)$, given in Proposition 3.4, and the Alexander-Fox matrix of $L \subset L(p, q)$ is given by $A_L(t) = A(t^{p'}, t^{q'})$. By deleting the last column of $A(x, a)$ (belonging to the lens relation), we obtain the two-variable Alexander-Fox matrix of the link

$$L' \cup U \subset S^3.$$

$$A_L(t) = \begin{matrix} & w_1 & \dots & w_{n-k} & q_1 & \dots & q_k & r_1 & \dots & r_k & l \\ \begin{matrix} x_n \\ \vdots \\ x_{k+1} \\ x_k \\ \vdots \\ x_1 \\ a_1 \\ \vdots \\ a_k \end{matrix} & \left(\begin{matrix} & & & & & & & & & & 0 \\ & & & & & & & & & & \vdots \\ & & & & & & & & & & 0 \\ & & & & & & & & & & \beta_k \\ & & & & & A_{L' \cup U}(t^{p'}, t^{q_{k'}}) & & & & & \vdots \\ & & & & & & & & & & \beta_1 \\ & & & & & & & & & & \alpha \\ & & & & & & & & & & 0 \\ & & & & & & & & & & 0 \end{matrix} \right) \end{matrix}$$

where $\alpha = \frac{\partial l}{\partial a_1}(t^{p'}, t^{q_{k'}})$ and $\beta_i = \frac{\partial l}{\partial x_i}(t^{p'}, t^{q_{k'}})$ for $i = 1, \dots, k$. Denote by $A^{i,(j,j')}$ the matrix we get from A by deleting the i -th row, the j -th and the j' -th column, and denote by $B = A_{L' \cup U}(t^{p'}, t^{q_{k'}})$. The Alexander polynomial of L may be written as

$$\begin{aligned} \Delta(L) &= \gcd\{\det A_L^{i,(j,j')} \mid 1 \leq i \leq n+k, 1 \leq j < j' \leq n+k+1\} = \\ &= \gcd\{\det B^{i,j}, \alpha \det B^{(i,n+1),(j,j')} + \sum_{r=1}^k (-1)^r \beta_r \det B^{(i,n-r+1),(j,j')} \mid 1 \leq i, j, j' \leq n+k\}. \end{aligned}$$

Since $\Delta_1(L' \cup U)(t^{p'}, t^{q_{k'}}) = \gcd\{B^{i,j} \mid 1 \leq i, j \leq n+k\}$ and $\Delta_2(L' \cup U)(t^{p'}, t^{q_{k'}}) = \gcd\{B^{(i,i'),(j,j')} \mid 1 \leq i, i', j, j' \leq n+k\}$, it follows that $\Delta_2(L' \cup U)(t^{p'}, t^{q_{k'}})$ divides $\Delta(L)$ and $\Delta(L)$ divides $\Delta_1(L' \cup U)(t^{p'}, t^{q_{k'}})$. \square

As we have seen, the group of a link $L \subset L(p, q)$ is in a close relationship with the group of the classical link $L' \cup U \subset S^3$. Based on this relationship, the Proposition 4.2 approximates the Alexander polynomial $\Delta(L)$ with the link polynomials of $L' \cup U$. Now we would like to describe the relation between the Alexander polynomials $\Delta(L)$ and $\Delta(L' \cup U)$ more precisely.

Definition 4.3. Let p and q be positive coprime integers. Denote by $\lambda_{p,q}$ the rational function, given by

$$\lambda_{p,q}(u) = \frac{u^{pq} - 1}{(u^p - 1)(u^q - 1)}.$$

Lemma 4.4. For any positive coprime integers p and q we have $\lambda_{p,q}(u) = \frac{\lambda_1(p,q)(u)}{u-1}$, where $\lambda_1(p, q)$ is a polynomial, not divisible by $(u-1)$.

Proof. In the rational function $\lambda_{p,q}$, every root of the denominator $(u^p - 1)(u^q - 1)$ is either the p -th or the q -th root of unity and thus is also a root of the numerator $(u^{pq} - 1)$. Since p and q are coprime, only 1 is at the same time the p -th and the q -th root of unity and thus cannot be cancelled. \square

Theorem 4.5. *Let L be a link in $L(p, q)$, which intersects the disk D in k transverse intersection points, so that $\bar{k} = \sum_{i=1}^k \epsilon_i \neq 0$. Let $p' = \frac{p}{d}$ and $k' = \frac{\bar{k}}{d}$ where $d = \gcd\{p, \bar{k}\}$. Then the Alexander polynomial of L and the two-variable Alexander polynomial of the classical link $L' \cup U$ are related by*

$$\Delta(L)(t) = \frac{\Delta(L' \cup U)(t^{p'}, t^{qk'})}{t^{k'} - 1}.$$

Proof. Represent L by a diagram, described in 2.9. Then the group of L has the presentation

$$\pi_1(L(p, q) \setminus L, *) = \langle x_1, \dots, x_n, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k, w_1, \dots, w_{n-k}, l \rangle,$$

described in Subsection 3.2. By calculating the Fox differentials of this presentation and then identifying all the x_i -variables and all the a_j -variables, we obtain the matrix $A(x, a)$. In the Proposition 3.4 we have shown that the Alexander-Fox matrix of L is calculated from $A(x, a)$ by the substitution $A_L(t) = A(t^{p'}, t^{qk'})$.

For $i = 1, \dots, k$, we denote: $\alpha = 1 + a + \dots + a^{p-1}$,

$$\beta_i = \begin{cases} -(1 + x^{\bar{k}} + \dots + x^{(q-1)\bar{k}})x^{\sum_{j=1}^{i-1} \epsilon_j} & \text{if } \epsilon_i = 1, \\ (1 + x^{\bar{k}} + \dots + x^{(q-1)\bar{k}})x^{\sum_{j=1}^i \epsilon_j} & \text{if } \epsilon_i = -1. \end{cases}$$

and

$$\phi_i = \begin{cases} 1 & \text{if } \epsilon_i = 1, \\ -x^{-1} & \text{if } \epsilon_i = -1. \end{cases}$$

and then calculate the matrix $A(x, a) =$

(2)

$$\begin{matrix} & w_1 & \dots & w_{n-k} & q_k & \dots & q_1 & r_k & \dots & r_1 & l \\ \begin{matrix} x_n \\ \vdots \\ x_{2k+1} \\ x_{2k} \\ \vdots \\ x_{k+1} \\ x_k \\ \vdots \\ x_1 \\ a_1 \\ \vdots \\ a_k \end{matrix} & \left(\begin{array}{cccccccccc} & & & & & & & & & & \\ & B_1 & & & 0 & & & & 0 & & 0 \\ & & & \phi_k a & & & & & & & \\ & B_3 & & & \ddots & & & & 0 & & 0 \\ & & & & & & \phi_1 a & & & & \\ & & & -\phi_k & & & & \phi_k(1-a) & & & \beta_k \\ & B_2 & & & \ddots & & & & \ddots & & \vdots \\ & & & & & & -\phi_1 & & & \phi_1(1-a) & \beta_1 \\ & & & \phi_k(1-x) & \dots & \phi_1(1-x) & x^{\epsilon_k} & & -1 & & \alpha \\ & 0 & & & 0 & & -1 & \ddots & & & 0 \\ & & & & & & & -1 & & x^{\epsilon_1} & \end{array} \right) \end{matrix}$$

If we erase the last column of A , corresponding to the lens relation, we obtain the (two variable) Alexander matrix of the classical link $L' \cup U$. The last column may be written as a linear combination of the columns r_1, \dots, r_k as follows:

$$(3) \quad l = \lambda_{p,q}(t^{k'}) \left((x^{\sum_{i=1}^{k-1} \epsilon_i}) r_k + (x^{\sum_{i=1}^{k-2} \epsilon_i}) r_{k-1} + \dots + (x^{\epsilon_1}) r_2 + r_1 \right).$$

We may use this linear combination when calculating the determinants of the minors of A_L . The Alexander polynomial of L is given by

$$\Delta(L) = \gcd\{\det A_L^{i,(j,j')} \mid 1 \leq i \leq n+k, 1 \leq j < j' \leq n+k+1\}.$$

We observe the minors $A_L^{i,(j,j')}$ and compare them with the minors of $A_{L' \cup U}$. Denote $B(t) = A_{L' \cup U}(t^{p'}, t^{q'})$. For $j' = n+k+1$ we have $\det A_L^{i,(j,n+k+1)} = \det B^{i,j}$.

The minor $A_L^{i,(j,j')}$ for $j' \leq n+k$ is obtained from the minor $B^{i,j}$ by changing the j' -th column to l and then subtracting from l all the possible terms in the linear combination (3) (we may subtract the terms whose corresponding columns are different from j and j').

For $1 \leq j < j' \leq n$ we have $\det A_L^{i,(j,j')} = 0$, since the last $(k+1)$ columns of this minor are linearly dependent.

For $j \leq n$ and $j' = n+r$ we have $\det A_L^{i,(j,n+r)} = \lambda_{p,q}(t^{k'}) x^{\sum_{i=1}^{k-r} \epsilon_i} \det B^{i,j}$ if $1 \leq r \leq k$.

For the remaining case when $n+1 \leq j < j' \leq n+r$, we argue as follows. If $i \leq n$, then the rows a_2, \dots, a_k are linearly dependent since each of them has at most $(k-2)$ nontrivial entries. If, on the other hand, $i > n$, then we have

$$\det A_L^{i,(j,j')} = \lambda_{p,q}(t^{k'}) \left(x^{\sum_{i=1}^{k-(j-n)} \epsilon_i} \det B^{i,j'} + x^{\sum_{i=1}^{k-(j'-n)} \epsilon_i} \det B^{i,j} \right).$$

Thus, we may write

$$\det A_L^{i,(j,j')} = \begin{cases} 0, & 1 \leq j < j' \leq n, \\ \det B^{i,j}, & j' = n+k+1, \\ \lambda_{p,q}(t^{k'}) x^{\sum_{i=1}^{k-r} \epsilon_i} \det B^{i,j}, & j \leq n, j' = n+r, 1 \leq r \leq k, \\ 0, & i \leq n, n+1 \leq j < j' \leq n+k, \\ \lambda_{p,q}(t^{k'}) \left(x^{\sum_{i=1}^{k-(j-n)} \epsilon_i} \det B^{i,j'} + x^{\sum_{i=1}^{k-(j'-n)} \epsilon_i} \det B^{i,j} \right), & i > n, n+1 \leq j < j' \leq n+k. \end{cases}$$

Since the Alexander polynomial is only defined up to the multiplication by a power of t , it follows that

$$\begin{aligned} \Delta(L)(t) &= \gcd\{\det A_L^{i,(j,j')} \mid 1 \leq i \leq n+k, 1 \leq j < j' \leq n+k+1\} = \\ &= \frac{\Delta(L' \cup U)(t^{p'}, t^{q'})}{t^{k'} - 1} \end{aligned}$$

by the Lemma 4.4. □

Corollary 4.6. *Let L be a link in $L(p, q)$, which intersects the disk D in k transverse intersection points, so that $\bar{k} = \sum_{i=1}^k \epsilon_i = 0$. Then the Alexander polynomial of L and the two-variable Alexander polynomial of the classical link $L' \cup U$ are related by*

$$\Delta(L)(t) = \frac{\Delta(L' \cup U)(t, t^q)}{t - 1}.$$

Proof. Represent L by a diagram, described in 2.9. By applying the generalized Reidemeister move $SL_{p,q}$ (see Theorem 2.5), we may change L to an equivalent link whose algebraic intersection with D equals $\bar{k} = \sum_{i=1}^k \epsilon_i = p$. Then we have $\gcd\{p, \bar{k}\} = p$, $p' = \frac{p}{d} = 1$ and $k' = \frac{\bar{k}}{d} = 1$. We proceed as in the proof of Theorem 4.5. \square

Corollary 4.7. *Let $L \subset L(p, q)$ be a link, which intersects the disk D in k transverse intersection points, so that $\bar{k} = 0$. Then the μ -twisted Alexander polynomial of L and the two-variable Alexander polynomial of the classical link $L' \cup U$ are related by*

$$\Delta^\mu(L)(t) = \frac{\Delta(L' \cup U)(t, \mu)}{\mu - 1},$$

where $\mu \in \mathbb{C}$ is any p -th complex root of unity, different from 1.

Proof. We use the same method as in the proof of the Theorem 4.5. In this case, $d = \gcd\{p, \bar{k}\} = p$ and therefore $p' = 1$ and $k' = 0$. The μ -twisted Alexander-Fox matrix of L is calculated from $A(x, a)$ by the substitution $A_L^\mu(t) = A(t, \mu)$ (see Proposition 3.4). For $i = 1, \dots, k$ we calculate: $\alpha = 1 + a + \dots + a^{p-1} = \frac{\mu^p - 1}{\mu - 1} = 0$ and

$$\beta_i = \begin{cases} -qx^{\sum_{j=1}^{i-1} \epsilon_j} & \text{if } \epsilon_i = 1, \\ qx^{\sum_{j=1}^i \epsilon_j} & \text{if } \epsilon_i = -1. \end{cases}$$

The matrix $A(x, a)$ looks like (2). If we erase the last column of $A(x, a)$, corresponding to the lens relation, we obtain the (two variable) Alexander matrix of the classical link $L' \cup U$. The last column may be written as a linear combination of the columns r_1, \dots, r_k as follows:

$$(4) \quad l = \frac{q}{1 - \mu} \left((x^{\sum_{i=1}^{k-1} \epsilon_i}) r_k + (x^{\sum_{i=1}^{k-2} \epsilon_i}) r_{k-1} + \dots + (x^{\epsilon_1}) r_2 + r_1 \right).$$

Denote $B(t) = A_{L' \cup U}(t, \mu)$. Using an analogous reasoning as in the proof of Theorem 4.5, we calculate

$$\det A_L^{i, (j, j')} = \begin{cases} 0, & 1 \leq j < j' \leq n, \\ \det B^{i, j}, & j' = n + k + 1, \\ \frac{q}{1 - \mu} x^{\sum_{i=1}^{k-r} \epsilon_i} \det B^{i, j}, & j \leq n, j' = n + r, 1 \leq r \leq k, \\ 0, & i \leq n, n + 1 \leq j < j' \leq n + k. \\ \frac{q}{1 - \mu} \left(x^{\sum_{i=1}^{k-(j-n)} \epsilon_i} \det B^{i, j'} + x^{\sum_{i=1}^{k-(j'-n)} \epsilon_i} \det B^{i, j} \right), & i > n, n + 1 \leq j < j' \leq n + k. \end{cases}$$

Since the Alexander polynomial is only defined up to multiplication by a power of t , it follows that

$$\Delta^\mu(L)(t) = \frac{\Delta(L' \cup U)(t, \mu)}{\mu - 1}.$$

□

Corollary 4.8. *Denote by $-L$ the link $L \subset L(p, q)$ with the opposite orientation. Then the Alexander polynomials of L and $-L$ are connected by*

$$\Delta(-L)(t) = \Delta(L)(t^{-1}).$$

Proof. By the Theorem 4.5, the Alexander polynomial of L is given by $\Delta(L)(t) = \frac{\Delta_{L' \cup U}(t^{p'}, t^{qk'})}{t^{k'} - 1}$. When changing the orientation of L , every sign ϵ_i switches to $-\epsilon_i$, thus $\bar{k} = \sum_{i=1}^k \epsilon_i$ becomes $-\bar{k}$ and consequently k' becomes $-k'$. The classical link L' , corresponding to L , also changes orientation, and for the classical links we know that $\Delta(-L')(t) = \Delta(L')(t^{-1})$. Therefore, by the Theorem 4.5 we calculate

$$\Delta(-L)(t) = \frac{\Delta(-L' \cup U)(t^{p'}, t^{-qk'})}{t^{-k'} - 1} = \frac{\Delta(L' \cup U)(t^{-p'}, t^{-qk'})}{t^{-k'} - 1} = \Delta(L)(t^{-1}).$$

□

Now we observe a family of unlinks $\{L_{k,r}\}_{k \in \mathbb{N}, 0 \leq r \leq k}$ in the lens space $L(p, q)$. Any link $L \subset L(p, q)$ may be obtained by combining ("multiply" connect summing) the classical link $L' \subset S^3$ with a suitable link $L_{k,r}$. We calculate the Alexander polynomial $\Delta(L_{k,r})$ for any $k \in \mathbb{N}, 0 \leq r \leq k$. In the following proposition, we begin with the subfamily $\{L_k\}_{k \in \mathbb{N}}$, where $L_k = L_{k,0}$.

Proposition 4.9. *Let k be a positive integer. Denote by $L_k \subset L(p, q)$ the unlink, consisting of k unknots, each of them intersecting the disk D transversely in a single positive point of intersection. The Alexander polynomial of L_k is given by*

$$\Delta(L_k)(t) = \frac{(t^{qk'} - 1)^{k-1}(t - 1)}{t^{k'} - 1}.$$

Proof. We will calculate the Alexander polynomial of the classical link $L'_k \cup U \subset S^3$ and then use the Theorem 4.5. Denote by x_i the generator of $\pi_1(S^3 \setminus (L'_k \cup U), *)$, corresponding to the i -th unknot of the unlink L_k for $i = 1, \dots, k$. Denote by a_1, \dots, a_k the generators, corresponding to the meridian U , so that the Wirtinger relations of the crossings between the meridian and the unknots are $q_i: a_1 x_i a_1^{-1} x_i^{-1}$ and $r_i: x_i a_{k-i+1} x_i^{-1} a_{(k-i+2) \bmod k}$ for $i = 1, \dots, k$. From the link group presentation

$$\pi_1(S^3 \setminus (L'_k \cup U), *) = \langle x_1, \dots, x_k, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k \rangle$$

we derive the Alexander matrix $A_{L'_k \cup U}$. For $k \geq 2$, we have

$$A_{L'_k \cup U}(x, a) = \begin{matrix} & q_k & \dots & q_1 & r_k & \dots & r_1 \\ \begin{matrix} x_k \\ \vdots \\ x_1 \\ a_1 \\ \vdots \\ a_k \end{matrix} & \begin{pmatrix} (a-1) & & & (1-a) & & \\ & \ddots & & & \ddots & \\ & & (a-1) & & & (1-a) \\ (1-x) & \dots & (1-x) & x & & -1 \\ & 0 & & -1 & \ddots & \\ & & & & -1 & x \end{pmatrix} \end{matrix}$$

Observe that this matrix is obtained from the matrix $A(x, a)$ in (1) if $n = k$, by adding the row x_{k+i} to x_i for $i = 1, \dots, k$ and then deleting the rows x_{k+1}, \dots, x_{2k} .

The Alexander polynomial is then calculated as

$$\Delta(L'_k \cup U)(x, a) = \gcd\{\det A_{L'_k \cup U}(x, a)^{i,j} \mid 1 \leq i, j \leq 2k\}.$$

where $A^{i,j}$ denotes the minor, obtained by deleting the i -th row and the j -th column of a matrix A . Denote by $S(i, j)$ the set of all bijections

$$\pi: \{1, \dots, \widehat{i}, \dots, 2k\} \rightarrow \{1, \dots, \widehat{j}, \dots, 2k\},$$

and let $m_{i,j}$ be the (i, j) -th entry of the matrix $A_{L'_k \cup U}$. Then we have

$$\det A_{L'_k \cup U}^{i,j} = \sum_{\pi \in S(i,j)} m_{1,\pi(1)} m_{2,\pi(2)} \dots m_{2k,\pi(2k)}.$$

From the matrix, we may observe that for every $\pi \in S(i, j)$, the following holds.

$$m_{1,\pi(1)} \dots m_{k,\pi(k)} = \begin{cases} 0 & \text{or} \\ \pm(a-1)^{k-1} & \text{if } i \leq k, \\ \pm(a-1)^k & \text{if } i \geq k+1. \end{cases}$$

$$m_{k+1,\pi(k+1)} \dots m_{2k,\pi(2k)} = \begin{cases} 0 & \text{or} \\ (1-x)(\dots) \text{ or } x^k \text{ or } -1 & \text{if } i \leq k \\ \text{something} & \text{if } i \geq k+1 \end{cases}$$

If $i \geq k+1$, then $\det A_{L'_k \cup U}^{i,j}$ is divisible by $(a-1)^k$. If $i \leq k$ and $j \geq k+1$, then $\det A_{L'_k \cup U}^{i,j} = (a-1)^{k-1}(1-x)(-1)^{j-k-1}x^{2k-j}$. If $i, j \leq k$, then we may expand the determinant along the $(k+1)$ -st row to obtain

$$\det A_{L'_k \cup U}^{i,j} = (a-1)^{k-1} \left(\pm(x^k - 1) + (1-x) \sum_{r=1}^k C_r \right),$$

where the factors C_r correspond to sums of the terms $m_{k+2,\pi(k+2)} \dots m_{2k,\pi(2k)}$.

It follows that the determinant $\det A_{L'_k \cup U}^{i,j}$ for $1 \leq i, j \leq 2k$ is always divisible by $(a-1)^{k-1} \gcd\{a-1, x-1\}$. Since $\det A_{L'_k \cup U}^{1,2k}(x, a) = (a-1)^{k-1}(1-x)(-1)^{k-1}$ and $\det A_{L'_k \cup U}^{k+1,2k}(x, a) = (a-1)^k(-1)^{k-1}$, we conclude that

$$\Delta(L'_k \cup U)(x, a) = (a-1)^{k-1} \gcd\{a-1, x-1\} .$$

By the Theorem 4.5 it follows

$$\Delta(L_k)(t) = \frac{(t^{qk'} - 1)^{k-1}}{t^{k'} - 1} \gcd\{t^{qk'} - 1, t^{p'} - 1\} = \frac{(t^{qk'} - 1)^{k-1}(t-1)}{t^{k'} - 1} .$$

Here we have used the fact that the numbers p' and qk' are coprime, and thus 1 is the only common root of the polynomials $t^{qk'} - 1$ and $t^{p'} - 1$.

For $k = 1$ we have

$$A_{L'_1 \cup U}(x, a) = \begin{matrix} & q_1 & r_1 \\ x_1 & \begin{pmatrix} a-1 & 1-a \\ 1-x & x-1 \end{pmatrix} \\ a_1 & \end{matrix}$$

and $\Delta(L'_1 \cup U)(t^{p'}, t^q) = \gcd\{t^q - 1, t^{p'} - 1\} = t - 1$ and it follows from the Theorem 4.5 that $\Delta(L_1) = 1$. \square

Corollary 4.10. *Let $L_{k,r} \subset L(p, q)$ be the unlink, consisting of k unknots, the i -th unknot intersecting the disk D transversely in a single point with the sign of intersection ϵ_i so that $\sum_{i=1}^k \epsilon_i = k - 2r$. The Alexander polynomial of $L_{k,r}$ is given by*

$$\Delta(L_{k,r})(t) = \frac{(t^{qk'} - 1)^{k-1}(t-1)}{t^{k'} - 1} .$$

Proof. We use the same notation as in the proof of the Proposition 4.9. Observe that r is the number of negative intersection points between $L_{k,r}$ and D . Since the components of $L_{k,r}$ are homeomorphic, we may assume that $\epsilon_i = -1$ for $i = 1, \dots, r$ and $\epsilon_j = 1$ for $j = r+1, \dots, k$. The Wirtinger relations of the crossings between the meridian and the unknots are $q_i: a_1 x_i^{\epsilon_i} a_1^{-1} x_i^{-\epsilon_i}$ and $r_i: x_i^{\epsilon_i} a_{k-i+1} x_i^{-\epsilon_i} a_{(k-i+2) \bmod k}$ for $i = 1, \dots, k$. From the link group presentation

$$\pi_1(S^3 \setminus (L'_{k,r} \cup U), *) = \langle x_1, \dots, x_k, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k \rangle$$

we derive the Alexander matrix $A_{L'_{k,r} \cup U}$. For $k \geq 2$, we have

$$A_{L'_{k,r} \cup U}(x, a) = \begin{matrix} & q_k & \dots & q_1 & r_k & \dots & r_1 \\ \begin{matrix} x_k \\ \vdots \\ x_1 \\ a_1 \\ \vdots \\ a_k \end{matrix} & \begin{pmatrix} \phi_k(a-1) & & & \phi_k(1-a) & & \\ & \ddots & & & \ddots & \\ & & \phi_1(a-1) & & & \phi_1(1-a) \\ \phi_k(1-x) & \dots & \phi_1(1-x) & x^{\epsilon_k} & & -1 \\ & & 0 & -1 & \ddots & \\ & & & -1 & & x^{\epsilon_1} \end{pmatrix} \end{matrix},$$

where

$$\phi_i = \begin{cases} -x^{-1} & \text{for } 1 \leq i \leq r, \\ 1 & \text{for } r+1 \leq i \leq k. \end{cases}$$

Observe that $\prod_{i=1}^k \phi_i = (-1)^r x^{-r}$ and $\bar{k} = \sum_{i=1}^k \epsilon_i = k - 2r$. The Alexander polynomial is calculated as

$$\Delta(L'_{k,r} \cup U)(x, a) = \gcd\{\det A_{L'_{k,r} \cup U}(x, a)^{i,j} \mid 1 \leq i, j \leq 2k\}.$$

If $i \geq k+1$, then $\det A_{L'_{k,r} \cup U}^{i,j}$ is divisible by $(-1)^r x^{-r} (a-1)^k$. If $i \leq k$ and $j \geq k+1$, then $\det A_{L'_{k,r} \cup U}^{i,j} = (a-1)^{k-1} (1-x) (-1)^{r+j-k-1} x^{\sum_{m=1}^{2k-j} \epsilon_m - r}$. If $i, j \leq k$, then we may expand the determinant along the $(k+1)$ -st row to obtain

$$\det A_{L'_{k,r} \cup U}^{i,j} = \begin{cases} (-1)^{r-1} x^{-r+1} (a-1)^{k-1} \left(\pm(x^{\bar{k}} - 1) + (1-x) \sum_{m=1}^k C_m \right) & \text{if } i = j \leq r, \\ (-1)^r x^{-r} (a-1)^{k-1} \left(\pm(x^{\bar{k}} - 1) + (1-x) \sum_{m=1}^k C_m \right) & \text{otherwise.} \end{cases}$$

for some factors C_m .

It follows that the determinant $\det A_{L'_{k,r} \cup U}^{i,j}$ for $1 \leq i, j \leq 2k$ is always divisible by $(a-1)^{k-1} \gcd\{a-1, x-1\}$. Since $\det A_{L'_{k,r} \cup U}^{1,2k}(x, a) = (a-1)^{k-1} (1-x) (-1)^{k+r-1} x^{-r}$ and $\det A_{L'_{k,r} \cup U}^{k+1,2k}(x, a) = (a-1)^k (-1)^{k+r-1} x^{-r}$, we conclude that

$$\Delta(L'_{k,r} \cup U)(x, a) = (a-1)^{k-1} \gcd\{a-1, x-1\}.$$

We have not taken into account the factors x^{-r} , since the Alexander polynomial is only defined up to the multiplication by a power of t . By the Theorem 4.5 it follows

$$\Delta(L_{k,r})(t) = \frac{\Delta(L'_{k,r} \cup U)(t^{p'}, t^{qk'})}{t^{k'} - 1} = \frac{(t^{qk'} - 1)^{k-1} (t - 1)}{t^{k'} - 1}.$$

We have used the fact that p' and qk' are coprime numbers, and thus 1 is the only common root of the polynomials $t^{qk'} - 1$ and $t^{p'} - 1$. \square

Theorem 4.11. *Let $L \subset L(p, q)$ be a link which intersects the disk D transversely in a single point of intersection. Denote by $L' \subset S^3$ the link with the same diagram as L , which we get by ignoring the surgery along the unknot U . Then the Alexander polynomials of L and L' are related by*

$$\Delta(L)(t) = \Delta(L')(t^p) .$$

Proof. In this case, $k = 1$ and $H_1(L(p, q) \setminus L)$ contains no torsion. Take the strand of L which intersects the disk D , and cut it on each side of D to represent L as the connected sum $L' \# L_1$ (L_1 is the unlink with one component, defined in the Proposition 4.9). We use the result [2, Proposition 8], which says the Alexander polynomial of a connected sum of two links in $L(p, q)$, one of them a local link, equals the product of both Alexander polynomials. Let $\pi_1(S^3 \setminus L', *) = \langle y_1, \dots, y_n \mid w_1, \dots, w_n \rangle$ be the Wirtinger presentation of the group of L' and $\pi_1(L(p, q) \setminus L_1, *) = \langle x_1, a_1 \mid q_1, r_1, l \rangle$ be the presentation of the group of L_1 as defined in the proof of Proposition 4.9. Then

$$\pi_1(L(p, q) \setminus L, *) = \langle y_1, \dots, y_n, x_1, a_1 \mid w_1, \dots, w_n, q_1, r_1, l, x_1 = y_1 \rangle$$

is the presentation of the group of L . The Alexander matrix of L looks like

$$A_L(t) = \begin{pmatrix} & A_{L'}(t^p) & & 0 & & -1 \\ & & & & & 0 \\ & & & & & \vdots \\ & & & & & 0 \\ 0 & & & A_{L_1}(t) & & 1 \\ & & & & & 0 \end{pmatrix} ,$$

where $A_{L'}(t)$ and $A_{L_1}(t)$ are the Alexander matrices of L' and L_1 respectively. In the Alexander matrix for L , all the generators, corresponding to the overpasses of L (these are y_1, \dots, y_n and x_1) are identified and sent to t^p , while the generator a_1 is sent to t^q (see Proposition 3.4). The Alexander polynomial of L is calculated by

$$\Delta(L)(t) = \gcd\{\det A_L(t)^{i, (j, j', j'')} \mid 1 \leq i \leq n+2, 1 \leq j < j' < j'' \leq n+4\} .$$

For the indices $1 \leq i \leq n+2$ and $1 \leq j < j' < j'' \leq n+4$ we have

$$\det A_L(t)^{i, (j, j', j'')} = \begin{cases} \det A_{L'}(t^p)^{i, j} \cdot \det A_{L_1}(t)^{1, (j'-n, j''-n)} & \text{if } i \leq n, j \leq n \text{ and } n+1 \leq j', j'' \leq n+3, \\ -\det A_{L'}(t^p)^{1, j} \cdot \det A_{L_1}(t)^{i, (j'-n, j''-n)} & \text{if } i \geq n+1, j \leq n \text{ and } n+1 \leq j', j'' \leq n+3, \\ 0 & \text{otherwise.} \end{cases}$$

Denoting by $d_i(A)$ the greatest common divisor of all i -minors of a matrix A , we have $d_{n+1}(A_L(t)) = d_{n-1}(A_{L'}(t^p)) \cdot d_1(A_{L_1}(t))$ and consequently $\Delta(L)(t) = \Delta(L')(t^p) \cdot \Delta(L_1)(t)$. We have shown in 4.9 that $\Delta(L_1) = 1$ and thus $\Delta(L)(t) = \Delta(L')(t^p)$. \square

Corollary 4.12. *Let $L \subset L(p, q)$ be a link, intersecting the disk D transversely in a single point of intersection. Then for the corresponding link $L' \subset S^3$ we have $\Delta(L' \cup U)(t^p, t^q) = (t-1)\Delta(L')(t^p)$.*

Proof. In this case $k = 1$. By the Theorem 4.11 we have $\Delta(L)(t) = \Delta(L')(t^p)$, and by the Theorem 4.5 we conclude $\Delta(L' \cup U)(t^p, t^q) = (t - 1)\Delta(L')(t^p)$. \square

Theorem 4.13. *Let $L \subset L(p, q)$ be a link which intersects the disk D transversely in k intersection points. Denote by $L' \subset S^3$ the link with the same diagram as L , which we get by ignoring the surgery along the unknot U . Let $L_k \subset L(p, q)$ be the unlink, defined in the Corollary 4.10, which intersects D in the same way as L . If the Alexander matrix of L' has rank at least k , then the Alexander polynomial $\Delta(L)(t)$ is divisible by the*

$$\gcd \left\{ \Delta_r(L')(t^{p'}) \cdot \Delta_{(k+1)-r}(L_k)(t) \mid r = 1, \dots, k \right\} .$$

Proof. Denote by x_1, \dots, x_k the generators of $\pi_1(L(p, q) \setminus L_k, *)$, corresponding to the components of L_k , obtaining the presentation

$$\pi_1(L(p, q) \setminus L_k, *) = \langle x_1, \dots, x_k, a_1, \dots, a_k \mid q_1, \dots, q_k, r_1, \dots, r_k, l \rangle ,$$

with the relations q_i and r_i defined in the proof of Corollary 4.10. Let $\pi_1(S^3 \setminus L', *) = \langle y_1, \dots, y_n \mid w_1, \dots, w_n \rangle$ be the Wirtinger presentation, where the generators y_1, \dots, y_k correspond to those overpasses of L' which intersect the unknot U . We obtain L from L' by identifying $x_i = y_i$ for $i = 1, \dots, k$. The presentation for $\pi_1(L(p, q) \setminus L, *)$ is obtained by joining the generators and relations of the presentations for $\pi_1(L(p, q) \setminus L_k, *)$ and $\pi_1(S^3 \setminus L', *)$ and adding the relations $x_i = y_i$ for $i = 1, \dots, k$. The Alexander matrix for L , corresponding to this presentation, is

$$A_L(t) = \begin{pmatrix} & & & 1 & & \\ & A_{L'}(t^{p'}) & 0 & & \ddots & \\ & & & & & 1 \\ & & & -1 & & \\ 0 & & A_{L_k}(t) & & \ddots & \\ & & & & & -1 \end{pmatrix} .$$

The Alexander polynomial of L is calculated as

$$\Delta(L)(t) = \gcd \{ \det A_L(t)^{i, (j_1, \dots, j_{k+2})} \mid 1 \leq i \leq n+2k, 1 \leq j_1 < j_2 < \dots < j_{k+2} \leq n+3k+1 \} ,$$

where $A^{i, (j_1, \dots, j_{k+2})}$ denotes the minor, obtained by deleting the row i and the columns j_1, \dots, j_{k+2} from a matrix A .

For the multiindex (j_1, \dots, j_{k+2}) , let $1 \leq j_1 < \dots < j_{r_1} \leq n < j_{r_1+1} < \dots < j_{r_1+r_2} \leq n+2k+1 < j_{r_1+r_2+1} < \dots < j_{k+2} \leq n+3k+1$. Denote

$$\mathcal{M}(k, r) = \{ M \subset \{1, \dots, k\} \mid |M| = r \} .$$

If $r_1 + r_2 = k + 2$ (we do not delete any of the last k columns), then we have

$$\det A_L(t)^{i, (j_1, \dots, j_{k+2})} =$$

$$\begin{cases} \pm \sum_{M \in \mathcal{M}(k, r_1-1)} \det A_{L'}(t^{p'})^{M \cup \{i\}, (j_1, \dots, j_{r_1})} \cdot \det A_{L_k}(t)^{\{1, \dots, k\} \setminus M, (j_{r_1+1}, \dots, j_{r_1+r_2})} & \text{if } i \leq n, \\ \mp \sum_{M \in \mathcal{M}(k, r_1)} \det A_{L'}(t^{p'})^{M, (j_1, \dots, j_{r_1})} \cdot \det A_{L_k}(t)^{(\{1, \dots, k\} \cup \{i\}) \setminus M, (j_{r_1+1}, \dots, j_{r_1+r_2})} & \text{if } i > n, \\ 0 & \text{otherwise,} \end{cases}$$

and thus $\det A_L(t)^{i, (j_1, \dots, j_{k+2})}$ is divisible by $\Delta_{r_1}(L')(t^{p'}) \cdot \Delta_{k+1-r_1}(L_k)(t)$.

Otherwise, if $r_1+r_2 = z < k+2$, then $\det A_L(t)^{i, (j_1, \dots, j_{k+2})}$ is divisible by $\Delta_{r_1}(L')(t^{p'}) \cdot \Delta_{z-1-r_1}(L_k)(t)$, which is also divisible by $\Delta_{r_1}(L')(t^{p'}) \cdot \Delta_{k+1-r_1}(L_k)(t)$. Therefore, the Alexander polynomial $\Delta(L)(t)$ is always divisible by the

$$\gcd\{\Delta_r(L')(t^{p'}) \cdot \Delta_{(k+1)-r}(L_k)(t) \mid r = 1, \dots, k\}.$$

□

5. EXAMPLES AND APPLICATIONS

Example 5.1. Denote by $K_i \subset L(p, q)$ the trefoil knots in Figure 7. Note that K_i intersects the disk D in i transverse intersection points for $i = 0, 1, 2$. The corresponding classical trefoil knot $K' \subset S^3$ has $\Delta(K')(t) = t^2 - t + 1$ and $\Delta_2(K')(t) = 1$. By the Corollary 4.1, the (twisted) Alexander polynomial of K_0 equals

$$\Delta^1(K_0)(t) = p(t^2 - t + 1).$$

By the Theorem 4.11, the Alexander polynomial of K_1 equals $\Delta(K_1)(t) = t^{2p} - t^p + 1$. For $K_2 \in L(p, q)$, we calculate $\Delta(K'_2 \cup U)(x, a) = (x^3 + a) \gcd\{x-1, a-1\}$. It follows from the Theorem 4.5 that the Alexander polynomial of K_2 is given by

$$\Delta(K_2)(t) = \begin{cases} \frac{\Delta(K'_2 \cup U)(t^p, t^{2q})}{t^2 - 1} = \frac{(t^{3p} + t^{2q})(t-1)}{t^2 - 1} = \frac{t^{2q}(t^{3p-2q} + 1)}{t+1} & \text{if } p \text{ is odd,} \\ \frac{\Delta(K'_2 \cup U)(t^{p'}, t^q)}{t-1} = \frac{(t^{3p'} + t^q)(t-1)}{t-1} = t^q(t^{\frac{3p-2q}{2}} + 1) & \text{if } p \text{ is even.} \end{cases}$$

It might be interesting to check whether a similar formula defines the Alexander polynomial of a general torus knot in $L(p, q)$.

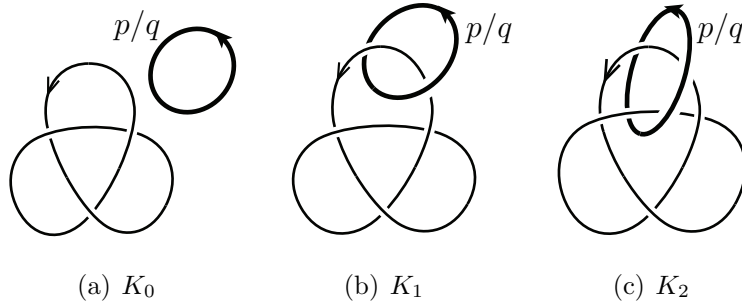


FIGURE 7. Diagrams of knots K_1 , K_2 , and K_3 .

Example 5.2. Observe the knot $K \subset L(4,1)$ in Figure 8. By the Corollary 2.13 we have $H_1(L(4,1) \setminus K) = \mathbb{Z} \oplus \mathbb{Z}_4$. The twisted Alexander polynomials of K are the following:

$$\begin{aligned}\Delta^1(K) &= 2t^{-2} - 2t^{-1} + 4 - 2t + 2t^2, \\ \Delta^{-1}(K) &= -2 + 2t, \\ \Delta^i(K) &= 1, \\ \Delta^{-i}(K) &= 1.\end{aligned}$$

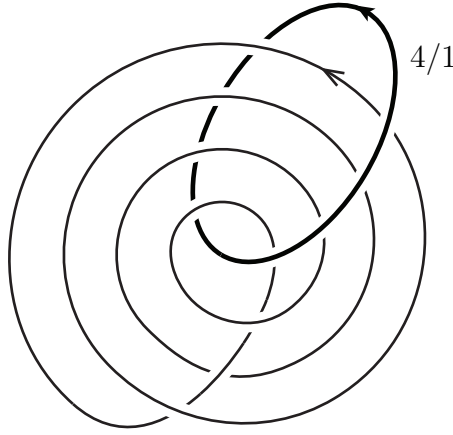


FIGURE 8. Diagrams of knots K_1 , K_2 , and K_3 .

Example 5.3. The Alexander polynomial is indeed a useful invariant that can detect inequivalent knots that are not distinguishable by other common invariants. Let K_1 and K_2 be knots in $L(3,1)$ with diagrams from Figure 9, these are respectively the links $L4a1$ and $L10n42$ from the Thistlethwaite link table with $3/1$ rational surgery performed on the trivial component. The knots are not distinguished by the Kauffman Bracket skein module $S_{2,\infty}$ [6], an invariant that generalized the kauffman bracket polynomial, since it holds that

$$S_{2,\infty}(K_1) = S_{2,\infty}(K_2) = -A^9 + A^4x + A.$$

Both knots are also homologous, $[K_1] = [K_2] \in H_1(L(p,1)) \cong \mathbb{Z}_3$, we can easily see that the complements have the same homology groups: $H_1(L(p,1) \setminus K_1) \cong H_1(L(p,1) \setminus K_2) \cong \mathbb{Z}$. However, the Alexander polynomial detects that they are indeed different knots:

$$\begin{aligned}\Delta_{K_1}(t) &= 1 \\ \Delta_{K_2}(t) &= t^{-6} - t^{-5} + t^{-2} - 1 + t^2 - t^5 + t^6.\end{aligned}$$

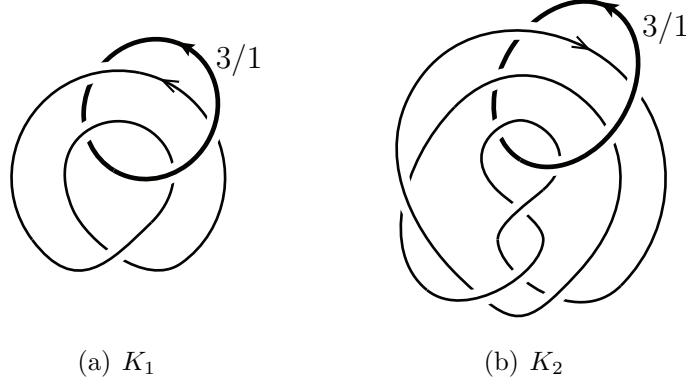


FIGURE 9. Two knots in $L(3, 1)$.

The relationship between the Alexander polynomial of links in the lens spaces and the classical Alexander polynomial we have obtained in Section 4 may be used to find the skein relation for the links in lens spaces.

Take a diagram of an oriented link $L \subset S^3$ and fix a crossing in the diagram. Denote L by L_+ if the chosen crossing is positive (see Definition 2.1), otherwise denote it by L_- . By switching the overpass and the underpass of the fixed crossing, the link L_+ becomes L_- and vice versa. By smoothing the fixed crossing, L becomes the link, denoted by L_0 . It is known that the Alexander polynomials of the three classical links L_+ , L_- and L_0 are related by the following **skein relation**:

$$\Delta(L_+) - \Delta(L_-) = (t^{\frac{1}{2}} - t^{-\frac{1}{2}})\Delta(L_0) .$$

Now take this setting into a lens space $L(p, q)$. Let $L \subset L(p, q)$ be an oriented link, given by a diagram, described in 2.9. We represent $L(p, q)$ as the $-p/q$ surgery on an oriented unknot U , and draw L inside this diagram. If D is the obvious disk in S^3 that is bounded by U , we denote by \bar{k} the algebraic intersection number of L and D . By the Lemma 2.11 we may assume that $0 \leq \bar{k} \leq p - 1$. We denote $p' = \frac{p}{d}$ and $k' = \frac{\bar{k}}{d}$, where $d = \gcd\{p, \bar{k}\}$. By the remark 3.5, p' may alternatively be defined as $p' = \frac{p}{\gcd\{p, \sum_{i=1}^r [L_i]\}}$, where $[L_i] \in H_1(L(p, q))$ denotes the homology class of the i -th component of L for $i = 1, \dots, r$.

Fix a crossing of the link L in the diagram (note that both strands of the crossing must belong to L , and not to the unknot U). Denote L by L_+ if the chosen crossing is positive, otherwise denote it by L_- . By switching the overpass and the underpass of the crossing, the link L_+ becomes L_- and vice versa. By smoothing the fixed crossing, L becomes the link, denoted by L_0 . The following Theorem describes the skein relation in the lens space $L(p, q)$.

Theorem 5.4. *Let $L \subset L(p, q)$ be a link with r components L_1, \dots, L_r . Denote by $[L_i] \in H_1(L(p, q))$ the homology class of the i -th component of L for $i = 1, \dots, r$ and let $p' = \frac{p}{\gcd\{p, \sum_{i=1}^r [L_i]\}}$. The Alexander polynomials of the three links $L_+, L_-, L_0 \subset$*

$L(p, q)$ are related by the skein relation

$$\Delta(L_+) - \Delta(L_-) = (t^{\frac{p'}{2}} - t^{-\frac{p'}{2}}) \Delta(L_0) .$$

Proof. Denote by L'_+ , L'_- and L'_0 the classical links we obtain from L_+ , L_- and L_0 by ignoring the surgery on the unknot U .

First let $\bar{k} \neq 0$. By the Theorem 4.5, the Alexander polynomials of L_+ , L_- and L_0 may be calculated from the 2-variable Alexander polynomials of the classical links $L'_+ \cup U$, $L'_- \cup U$ and $L'_0 \cup U$. We obtain

$$\begin{aligned} \Delta(L_+)(t) - \Delta(L_-)(t) &= \frac{\Delta(L'_+ \cup U)(t^{p'}, t^{qk'}) - \Delta(L'_- \cup U)(t^{p'}, t^{qk'})}{t^{k'} - 1} = \\ &= \left((t^{p'})^{\frac{1}{2}} - (t^{p'})^{-\frac{1}{2}} \right) \frac{\Delta(L'_0 \cup U)(t^{p'}, t^{qk'})}{t^{k'} - 1} = \left(t^{\frac{p'}{2}} - t^{-\frac{p'}{2}} \right) \Delta(L_0)(t) . \end{aligned}$$

Now let $\bar{k} = 0$; then $d = \gcd\{p, 0\} = p$ and $p' = 1$. By the Corollary 4.6, the Alexander polynomials of L_+ , L_- and L_0 may be calculated from the 2-variable Alexander polynomials of the classical links $L'_+ \cup U$, $L'_- \cup U$ and $L'_0 \cup U$, and we obtain

$$\begin{aligned} \Delta(L_+)(t) - \Delta(L_-)(t) &= \frac{\Delta(L'_+ \cup U)(t, t^q) - \Delta(L'_- \cup U)(t, t^q)}{t - 1} = \\ &= \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}} \right) \frac{\Delta(L'_0 \cup U)(t, t^q)}{t - 1} = \left(t^{\frac{p'}{2}} - t^{-\frac{p'}{2}} \right) \Delta(L_0)(t) . \end{aligned}$$

□

To conclude, we propose some open questions for a further consideration:

- Does the twisted Alexander polynomial of a link in any 3-manifold respect a skein relation?
- We know that the Alexander polynomial of a link in S^3 can be obtained as a special case of the HOMFLY-PT polynomial. Is this also the case for the HOMFLY-PT polynomial defined by Kalfagianni and Lin in [13]?
- How is the Alexander polynomial connected to the HOMFLY-PT skein module of $L(p, 1)$, calculated in [9]?

Acknowledgments. The second author was supported by the Slovenian Research Agency grant J1-7025.

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